

$SU(1, 1)$ *Nonlinear* Coherent States

B. Mojaveri* and A. Dehghani†

Department of Physics, Azarbaijan Shahid Madani University, PO Box 51745-406, Tabriz, Iran

Physics Department, Payame Noor University, PO Box 19395-4697, Tehran, Iran

January 1, 2013

Abstract

The idea of construction of the nonlinear coherent states based on the hypergeometric-type operators associated to the Weyl-Heisenberg group [*J.Phys.A* 45(2012) 095304], are generalized to the similar states for the arbitrary Lie group $SU(1, 1)$. By using of a discrete, unitary and irreducible representation of the Lie algebra $su(1, 1)$ wide range of generalized nonlinear coherent states (GNCS) have been introduced, which admit a resolution of the identity through positive definite measures on the complex plane. We have shown that realization of these states for different values of the deformation parameters $r = 1$ and 2 lead to the well-known Klauder-Perelomov and Barut-Girardello coherent states associated to the Irreps of the Lie algebra $su(1, 1)$, respectively. It is worth to mention that, like the canonical coherent states, GNCS possess the temporal stability property. Finally, studying some statistical characters implies that they have indeed nonclassical features such as squeezing, anti-bunching effect and sub-Poissonian statistics, too.

PACS Nos: 42.50.Dv, 03.65-w

1 Introduction

Coherent states, were first established by Schrodinger [1] as the eigenvectors of the boson annihilation operator, \hat{a} , corresponding to the Heisenberg-Weyl algebra. They play an important role in quantum optics and provide us with a link between quantum and classical oscillators. Moreover, these states can be produced by acting of the Glauber displacement operator, $D(z) = e^{z\hat{a}^\dagger - \bar{z}\hat{a}}$, on the vacuum states, where z is a complex variable. These states were later applied successfully to some other models based on their Lie algebra symmetries by Glauber [2, 3], Klauder [4, 5], Sudarshan [6], Barut and Girardello [7] and Perelomov [8]. Additionally, for the models with one degree of freedom either discrete or continuous spectra with no remark on the existence of a Lie algebra symmetry- Gazeau et al proposed new coherent states, which were parametrized by two real parameters [9, 10]. Moreover, there exist

*Email: bmojaveri@azaruniv.ac.ir

†Email: a_dehghani@tabrizu.ac.ir

some considerations in connection with coherent states corresponding to the shape invariance symmetries [11, 12]. To construct coherent states, four main different approaches the so-called Schrödinger, Klauder-Perelomov, Barut-Girardello, and Gazeau-Klauder methods have been found, so that the second and the third approaches rely directly on the Lie algebra symmetries and their corresponding generators. Here, it is necessary to emphasize that quantum coherence of states nowadays pervade many branches of physics such as quantum electrodynamics, solid-state physics, and nuclear and atomic physics, from both theoretical and experimental viewpoints.

In addition to CSs, squeezed states (SSs) are becoming increasingly important. These are the non-classical states of the electromagnetic field in which certain observables exhibit fluctuations less than in the vacuum state [13]. These states are important because they can achieve lower quantum noise than the zero-point fluctuations of the vacuum or coherent states. Over the last four decades there have been several experimental demonstrations of nonclassical effects, such as the photon anti-bunching [14], sub-Poissonian statistics [15, 16], and squeezing [17, 18]. On the other hand, considerable attention has been paid to the deformation of the harmonic oscillator algebra of creation and annihilation operators [19]. Some important physical concepts such as the CSs, the even- and odd-coherent states for ordinary harmonic oscillator have been extended to deformation case. Moreover, there exist interesting quantum effects, and related quantum states that are namely superposition states exhibiting quantum interference effects [20, 21]. Besides, superpositions of coherent states can be prepared in the motion of a trapped ion [22, 23]. With respect to the nonclassical effects, the coherent states turn out to define the limit between the classical and nonclassical behavior.

Another type of generalization of coherent states is the nonlinear coherent states (NLCSs), or f-CSs. These states correspond to nonlinear algebras rather than Lie algebras. The nonlinear coherent states NLCSs, $|z, f\rangle$, are right-hand eigenstates of the product of nonlinear function $f(\hat{N})$ of the number operator \hat{N} and the boson annihilation operator \hat{a} , i.e. they satisfy $f(\hat{N})\hat{a}|z, f\rangle = z|z, f\rangle$. The nature of the nonlinearity depends on the choice of the function $f(\hat{N})$ [24]. These states may appear as stationary states of the center-of mass motion of a trapped ion [25, 26]. NLCSs exhibit nonclassical features such as quadrature squeezing, sub-Poissonian statistics, anti-bunching, self-splitting effects and so on [27–30].

Our procedure is based on the idea of construction of the nonlinear coherent states in terms of the hypergeometric-type operators has already been established in Ref. [31] associated to the Weyl-Heisenberg group. In the present paper, we generalized it to create similar states for the arbitrary Lie group $SU(1, 1)$ corresponding to the Calogero-Sutherland model, which has attracted considerable interest [32]. The two-particle Calogero-Sutherland model enjoys the $su(1, 1)$ dynamic symmetry [33, 34] and its coherent states are investigated in [35, 36]. The $su(1, 1)$ Lie algebra is of great interest in quantum optics because it can characterize many kinds of quantum optical systems [7, 8, 37]. It has recently been used by many researchers to investigate the nonclassical properties of light in quantum optical systems [38]. In particular, the bosonic realization of $su(1, 1)$ describes the degenerate and non-degenerate parametric amplifiers [39]. The squeezed states of photons have been considered in terms of $su(1, 1)$ Lie algebra and the coherent states associated with this algebra. So, we decided to apply the theme had been discussed in [31] to investigate the $su(1, 1)$ nonlinear coherent states. By using of a discrete, unitary and irreducible representation of the Lie algebra

$su(1, 1)$ wide range of GNCSs have been introduced, which admit a resolution of the identity through positive definite and non-oscillating measures on the complex plane. As a concrete example we realized these states for different values of the deformation parameter $r = 1$ and 2 , which lead to the well-known Klauder-Perelomov and Barut-Girardello coherent states attached to the Calogero-Sutherland model, respectively [36]. Some interesting features are found. For instance, we have shown that they evolve in time as like as the canonical coherent states, in other words GNCS possess the temporal stability property. Furthermore, it has been discussed in detail that they have indeed nonclassical features such as squeezing, anti-bunching effect and sub-Poissonian statistics, too.

This paper is organized as follows: in section 2, we will bring a brief review on a way which results in a new kind of $su(1, 1)$ GNLCs, $|z\rangle_r^\lambda$. In order to realize the resolution of the identity, we have found the positive definite measures on the complex plane. With a review on these states, the relation between Klauder-Perelomov and Barut-Girardello coherent states of $su(1, 1)$ with nonlinear GNLCs will be obvious. It has been shown that these states can be considered as eigenstates of a certain annihilation operator, then they can be interpreted as NLCSs with a special nonlinearity function. Furthermore, section 3 is devoted to studying their non-classical properties.

2 Review and Construction

In Refs. [36, 40], it has been shown that the second-order differential operators

$$\begin{aligned} J_\pm^\lambda &:= \frac{1}{4} \left[\left(x \mp \frac{d}{dx} \right)^2 - \frac{\lambda(\lambda-1)}{x^2} \right], \\ J_3^\lambda &:= \frac{H^\lambda}{2} = \frac{1}{4} \left[-\frac{d^2}{dx^2} + x^2 + \frac{\lambda(\lambda-1)}{x^2} \right], \end{aligned} \quad (1)$$

satisfy the standard commutation relations of $su(1, 1)$ Lie algebra as follows

$$[J_+^\lambda, J_-^\lambda] = -2J_3^\lambda, \quad [J_3^\lambda, J_\pm^\lambda] = \pm J_\pm^\lambda. \quad (2)$$

Here, H^λ recalls the Calogero-Sutherland Hamiltonian on the half-line x and the simple an-harmonic term, $\frac{\lambda(\lambda-1)}{x^2}$, given in the Hamiltonian refers to the Goldman-Krivchenkov potential, too [41]. In terms of the Fock states, defined by the associated Laguerre polynomials [42] $L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \left(\frac{d}{dx} \right)^n (x^{n+\alpha} e^{-x})$ with $Re(\alpha) > -1$,

$$\langle x|n, \lambda\rangle = (-1)^n \sqrt{\frac{2\Gamma(n+1)}{\Gamma(n+\lambda+\frac{1}{2})}} x^\lambda e^{-\frac{x^2}{2}} L_n^{\lambda-\frac{1}{2}}(x^2), \quad \lambda > \frac{-1}{2}, \quad (3)$$

one can realize that the infinite dimensional Hilbert space $\mathcal{H}^\lambda := \text{span}\{|n, \lambda\rangle\}_{n=0}^\infty$ products the unitary and positive-integer irreps of $su(1, 1)$ Lie algebra as

$$J_+^\lambda |n-1, \lambda\rangle = \sqrt{n \left(n + \lambda - \frac{1}{2} \right)} |n, \lambda\rangle, \quad (4a)$$

$$J_-^\lambda |n, \lambda\rangle = \sqrt{n \left(n + \lambda - \frac{1}{2} \right)} |n-1, \lambda\rangle, \quad (4b)$$

$$J_3^\lambda |n, \lambda\rangle = \left(n + \frac{\lambda}{2} + \frac{1}{4} \right) |n, \lambda\rangle. \quad (4c)$$

It is straightforward that the orthogonality condition of the associated Laguerre polynomials lead to the following orthogonality condition of the basis of the Hilbert space \mathcal{H}^λ :

$$\langle n, \lambda | m, \lambda \rangle := \frac{2n!}{\Gamma(n + \lambda + \frac{1}{2})} \int_0^\infty x^{2\lambda} e^{-x^2} L_n^{\lambda-\frac{1}{2}}(x^2) L_m^{\lambda-\frac{1}{2}}(x^2) dx = \delta_{nm}. \quad (5)$$

It is useful to stress that the two operators J_+^λ and J_-^λ are Hermitian conjugate of each others with respect to the inner product (5) and J_3^λ is self-adjoint operator, too.

According to the definition has already been given in Ref. [31], the following GNCS are produced, here, via generalized analogue of the displacement operators acting on the vacuum state $|0, \lambda\rangle$

$$|z\rangle_r^\lambda := M_r^{\lambda-\frac{1}{2}}(|z|)_1 F_r \left(\left[\lambda - \frac{1}{2} \right], \left[\lambda - \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda - \frac{3}{2} + r \right], z J_+^\lambda \right) |0, \lambda\rangle, \quad r \geq 1, \quad (6)$$

where $z (= |z|e^{i\varphi})$ and r are respectively the coherence and the deformation parameters, respectively. Clearly, $|z\rangle_r^\lambda$ becomes the Klauder-Perelomov coherent states for the Calogero-Sutherland model, $|z\rangle_{KP}^\lambda$ (Eq. (11) in Ref. [36]), when r tends to unity and z be replaced with $\frac{z}{|z|} \tanh(|z|)$. Using the series form of the hypergeometric functions and applying the ladder relations, Eqs. (4), GNCS can be expanded into the basis $|n, k\rangle$ as

$$|z\rangle_r^\lambda = M_r^{\lambda-\frac{1}{2}}(|z|) \sum_{n=0}^\infty z^n \prod_{k=1}^{r-1} \frac{\Gamma(\lambda + k - \frac{1}{2})}{\Gamma(n + \lambda + k - \frac{1}{2})} \sqrt{\frac{\Gamma(n + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(n+1)}} |n, \lambda\rangle, \quad r \geq 2, \quad (7)$$

where $M_r^\lambda(|z|)$ is chosen so that $|z\rangle_r^\lambda$ is normalized, i.e. ${}_r\langle z|z\rangle_r^\lambda = 1$, then

$$M_r^\lambda(|z|) = {}_1F_{2r-2} \left(\left[\lambda + \frac{1}{2} \right], \left[\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda - \frac{3}{2} + r, \lambda - \frac{3}{2} + r \right], |z|^2 \right). \quad (8)$$

It should be noticed that, these states can be categorized as special class of *Generalized Hypergeometric Coherent States* [43] have already been made by Appl et al.

The expansion (7) also leads to the following non-vanishing phrases for the scalar products

$${}_r\langle z_1 | z_2 \rangle_r^\lambda = \frac{{}_1F_{2r-2} \left(\left[\lambda + \frac{1}{2} \right], \left[\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda - \frac{3}{2} + r, \lambda - \frac{3}{2} + r \right], \bar{z}_1 z_2 \right)}{\sqrt{M_r^\lambda(|z_1|) M_r^\lambda(|z_2|)}}, \quad (9)$$

$${}_{r_1}\langle z | z \rangle_{r_2}^\lambda = \frac{{}_1F_{r_1+r_2-2} \left(\left[\lambda + \frac{1}{2} \right], \left[\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda - \frac{3}{2} + r_1, \lambda - \frac{3}{2} + r_2 \right], |z|^2 \right)}{\sqrt{M_{r_1}^\lambda(|z|) M_{r_2}^\lambda(|z|)}}, \quad (10)$$

and result that two different kinds of these states are non-orthogonal, if $r_1 \neq r_2, z_1 \neq z_2$. Now, we are in a position to introduce the appropriate measure $d\mu_r(|z|) := K_r^\lambda(|z|) \frac{d|z|^2 d\varphi}{2}$ so that the resolution of the identity is realized for the coherent states $|z\rangle_r^\lambda$ in the Hilbert space \mathcal{H}^λ :

$$\begin{aligned} 1_{\mathcal{H}^\lambda} &= \oint_{\mathbb{C}(z)} |z\rangle_r^\lambda \langle z| d\mu_r(|z|) \\ &= 2\pi \sum_{n=0}^{\infty} \left[\prod_{k=1}^{r-1} \frac{\Gamma(\lambda + k - \frac{1}{2})}{\Gamma(n + \lambda + k - \frac{1}{2})} \right]^2 \frac{\Gamma(n + \lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1}{2})\Gamma(n + 1)} |n, \lambda\rangle \langle n, \lambda| \int_0^\infty |z|^{2n+1} \frac{K_r^\lambda(|z|)}{M_r^\lambda(|z|)} d|z|. \end{aligned} \quad (11)$$

It is found that using by the integral relation for the Meijers G-functions (see $\frac{7-811}{4}$ in [42]), these states resolve the unity operator for any r and λ through a positive definite and non-oscillating measure

$$\begin{aligned} K_r^\lambda(|z|) &= \frac{2\Gamma(\lambda + \frac{1}{2}) {}_1F_{2r-2} \left(\left[\lambda + \frac{1}{2} \right], \left[\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda - \frac{3}{2} + r, \lambda - \frac{3}{2} + r \right], |z|^2 \right)}{\pi \left[\prod_{k=1}^{r-1} \Gamma(\lambda + k - \frac{1}{2}) \right]^2} \\ &\quad \times G \left(\left[\begin{matrix} [] \\ [0, \lambda - \frac{1}{2}, \lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda - \frac{5}{2} + r, \lambda - \frac{5}{2} + r] \end{matrix} \right], |z|^2 \right). \end{aligned} \quad (12)$$

For $(\lambda; r) = (\frac{3}{4}; 1)$ as well as $(\lambda; r) = (\frac{3}{2}; 2, 3 \text{ and } 4)$ we have plotted the changes of the non-oscillating and positive definite measures $K_r^\lambda(|z|)$ in terms of $|z|^2$ in figures 1(a) and 1(b), respectively.

\diamond Coordinate Representation of $|z\rangle_r^\lambda$

Based on a new expression of the Laguerre polynomials as an operator-valued function given in [44]

$$L_n^\alpha(y) = \frac{1}{n!} y^{-\alpha} \left(\frac{d}{dy} - 1 \right)^n y^{n+\alpha}, \quad (13)$$

also, according to Eqs. (3) and (7) we have

$$\begin{aligned} \langle x|z\rangle_r^\lambda &= \sqrt{\frac{2}{M_r^\lambda(|z|)\Gamma(\lambda + \frac{1}{2})}} \sum_{n=0}^{\infty} (-z)^n \prod_{k=1}^{r-1} \frac{\Gamma(\lambda + k - \frac{1}{2})}{\Gamma(n + \lambda + k - \frac{1}{2})} x^\lambda e^{-\frac{x^2}{2}} L_n^{\lambda-\frac{1}{2}}(x^2) \\ &= \sqrt{\frac{2}{M_r^\lambda(|z|)\Gamma(\lambda + \frac{1}{2})}} \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \prod_{k=1}^{r-1} \frac{\Gamma(\lambda + k - \frac{1}{2})}{\Gamma(n + \lambda + k - \frac{1}{2})} e^{-\frac{y}{2}} y^{\frac{1-\lambda}{2}} \left(\frac{d}{dy} - 1 \right)^n y^{n+\lambda-\frac{1}{2}} \Big|_{y=x^2}. \end{aligned} \quad (14)$$

Along with substitution

$$\left(\frac{d}{dy} - 1 \right)^n y^n = \left(y \frac{d}{dy} + n - y \right) \dots \left(y \frac{d}{dy} + 1 - y \right) = \left(y \frac{d}{dy} - y + 1 \right)_n, \quad (15)$$

it becomes

$$\begin{aligned}
\langle x|z\rangle_r^\lambda &= \sqrt{\frac{2}{M_r^\lambda(|z|)\Gamma(\lambda + \frac{1}{2})}} e^{-\frac{y}{2}} y^{\frac{1-\lambda}{2}} \sum_{n=0}^{\infty} \frac{\left(y\frac{d}{dy} - y + 1\right)_n}{\prod_{k=1}^{r-1} \left(\lambda + k - \frac{1}{2}\right)_n} \frac{(-z)^n}{n!} y^{\lambda-\frac{1}{2}}|_{y=x^2} \\
&= \sqrt{\frac{2}{M_r^\lambda(|z|)\Gamma(\lambda + \frac{1}{2})y^{\lambda-1}e^y}} \\
&\quad \times {}_1F_{r-1}\left(\left[y\frac{d}{dy} - y + 1\right], \left[\lambda + \frac{1}{2}, \dots, \lambda - \frac{3}{2} + r\right], -z\right) y^{\lambda-\frac{1}{2}}|_{y=x^2}, \quad (16)
\end{aligned}$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, denotes the Pochhammer symbol. For instance, the explicit compact forms of $|z\rangle_2^\lambda$ and $|z\rangle_3^\lambda$ are

$$\begin{aligned}
\langle x|z\rangle_2^\lambda &= \left[\frac{2e^{-y}y^\lambda}{M_2^\lambda(|z|)\Gamma(\lambda + \frac{1}{2})}\right]^{\frac{1}{2}} {}_1F_1\left(\left[y\frac{d}{dy} - y + \lambda + \frac{1}{2}\right], \left[\lambda + \frac{1}{2}\right], -z\right) \mathbf{1}_{y=x^2} \\
&= \left[\left(-\frac{z}{|z|}\right)^{\frac{1}{2}-\lambda} \frac{2x}{I_{\lambda-\frac{1}{2}}(2|z|)}\right]^{\frac{1}{2}} e^{-z-\frac{x^2}{2}} J_{\lambda-\frac{1}{2}}(2ix\sqrt{z}), \quad (17a)
\end{aligned}$$

$$\begin{aligned}
\langle x|z\rangle_3^\lambda &= \left[\frac{2e^{-y}y^\lambda}{M_3^\lambda(|z|)\Gamma(\lambda + \frac{1}{2})}\right]^{\frac{1}{2}} {}_1F_2\left(\left[y\frac{d}{dy} - y + \lambda + \frac{1}{2}\right], \left[\lambda + \frac{1}{2}, \lambda + \frac{3}{2}\right], -z\right) \mathbf{1}_{y=x^2} \\
&= \frac{1}{\sqrt{{}_0F_3\left([\lambda + \frac{1}{2}, \lambda + \frac{3}{2}, \lambda + \frac{3}{2}], |z|^2\right)}} x^{2\lambda} e^{-x^2} {}_0F_1\left([\lambda + \frac{3}{2}], z(x^2 - 1)\right). \quad (17b)
\end{aligned}$$

As a consequence, one can check that for the case $r = 2$, we have

$$\begin{aligned}
|z\rangle_2^\lambda &= M_2^{\lambda-\frac{1}{2}}(|z|) {}_1F_2\left([\lambda + \frac{1}{2}], zJ_+^\lambda\right) |0, \lambda\rangle \\
&= M_2^{\lambda-\frac{1}{2}}(|z|) \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(n+1)\Gamma(n + \lambda + \frac{1}{2})}} |n, \lambda\rangle,
\end{aligned}$$

meanwhile it satisfies following eigenvalue equation

$$J_-^\lambda |z\rangle_2^\lambda = z|z\rangle_2^\lambda.$$

Then, $|z\rangle_{r \rightarrow 2}^\lambda$ is reduced to the $SU(1,1)$ coherency of the Barut-Girardello type, has already given in Ref [36] corresponding to the Calogero-Sutherland model.

◇ Time Evolution of $SU(1,1)$ GNCS

Due to the relations (1) and (4c), we have

$$H^\lambda |n, \lambda\rangle = (2n + \lambda + \frac{1}{2}) |n, \lambda\rangle. \quad (18)$$

Then the coherent states (7) evolve in time as

$$\begin{aligned} e^{-itH^\lambda} |z\rangle_r^\lambda &= \frac{e^{-it(\lambda+\frac{1}{2})}}{\sqrt{M_r^\lambda(|z|)}} \sum_{n=0}^{\infty} (ze^{-i2t})^n \prod_{k=1}^{r-1} \frac{\Gamma(\lambda+k-\frac{1}{2})}{\Gamma(n+\lambda+k-\frac{1}{2})} \sqrt{\frac{\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})\Gamma(n+1)}} |n, \lambda\rangle \\ &= e^{-it(\lambda+\frac{1}{2})} |ze^{-i2t}\rangle_r^\lambda, \end{aligned} \quad (19)$$

and confirm that these states are temporally stable.

3 Non-classical properties of $|z\rangle_r^\lambda$

In this section, we will set up detailed studies on statistical properties of GNCS. For this reason, proportional nonlinear function associated to them are introduced. Moreover, to analyze their statistical behavior, some of the characters including the second-order correlation function, Mandel's parameter and quadrature squeezing are computed.

\diamond Nonlinearity function

The question we pose now is whether the $su(1,1)$ coherent states constructed above can be defined as the eigenstates of the non-Hermitian and deformed annihilation operator $f(\hat{N})J_-^\lambda$, i.e.

$$f(\hat{N})J_-^\lambda |z\rangle_r^\lambda = z |z\rangle_r^\lambda, \quad (20)$$

where $f(\hat{N})$, is determined in terms of the number operator¹ $\hat{N} (= J_3^\lambda - \frac{\lambda}{2} - \frac{1}{4})$, plays an important role as a nonlinearity function [25]. Combining definition of the $su(1,1)$ nonlinear coherent states (7) and ladder relations (4), we get

$$\left[\frac{\Gamma(\hat{N} + \lambda + r - \frac{1}{2})}{\Gamma(\hat{N} + \lambda + \frac{3}{2})} \right] J_-^\lambda |z\rangle_r^\lambda = z |z\rangle_r^\lambda. \quad (21)$$

So $|z\rangle_r^\lambda$ can be identified as new classes of $su(1,1)$ GNCS with characterized nonlinearity functions, $\frac{\Gamma(\hat{N} + \lambda + r - \frac{1}{2})}{\Gamma(\hat{N} + \lambda + \frac{3}{2})}$. Obviously $f(\hat{N}) \rightarrow 1$ when $r \rightarrow 2$.

\diamond $SU(1,1)$ squeezing

We introduce two generalized Hermitian quadrature operators X_1 and X_2

$$X_1^\lambda = \frac{J_+^\lambda + J_-^\lambda}{2}, \quad X_2^\lambda = \frac{J_-^\lambda - J_+^\lambda}{2i}, \quad (22)$$

with the commutation relation $[X_1^\lambda, X_2^\lambda] = iJ_3^\lambda$. From this commutation relation the uncertainty relation for the variances of the quadrature operators X_i follows

$$\langle (\Delta X_1^\lambda)^2 \rangle \langle (\Delta X_2^\lambda)^2 \rangle \geq \frac{|\langle J_3^\lambda \rangle|^2}{4}, \quad (23)$$

¹ $\hat{N}|n, \lambda\rangle = n|n, \lambda\rangle$.

where $\langle(\Delta X_1^\lambda)^2\rangle = \langle(X_1^\lambda)^2\rangle - \langle X_1^\lambda\rangle^2$ and the angular brackets denote averaging over an arbitrary normalizable state for which the mean values are well defined, $\langle X_i\rangle = {}^\lambda_r\langle z|X_i|z\rangle_r^\lambda$. Following Walls (1983) as well as Wodkiewicz and Eberly (1985) [45, 39] we will say that the state is $SU(1, 1)$ squeezed if the condition

$$\langle(\Delta X_i^\lambda)^2\rangle < \frac{|\langle J_3^\lambda\rangle|}{2}, \quad \text{for } i = 1 \text{ or } 2, \quad (24)$$

is fulfilled. In other words, a set of quantum states are called squeezed states if they have less uncertainty in one quadrature (X_1 or X_2) than coherent states. To measure the degree of the $SU(1, 1)$ squeezing we introduce the squeezing factor S_i^λ [46]

$$S_i^\lambda = \frac{\langle(\Delta X_i^\lambda)^2\rangle - \frac{|\langle J_3^\lambda\rangle|}{2}}{\frac{|\langle J_3^\lambda\rangle|}{2}}, \quad (25)$$

it leads that the $SU(1, 1)$ squeezing condition takes on the simple form $S_i^\lambda < 0$, however maximally squeezing is obtained for $S_i^\lambda = -1$. By using of the mean values of the generators of the $SU(1, 1)$ Lie algebra, one can derive that uncertainty in the quadrature operators X_i can be expressed as the following forms

$$\langle(\Delta X_{1(2)}^\lambda)^2\rangle = \frac{2\langle J_+^\lambda J_-^\lambda\rangle + 2\langle J_3^\lambda\rangle \pm \langle J_+^{\lambda^2} + J_-^{\lambda^2}\rangle - \langle J_-^\lambda \pm J_+^\lambda\rangle^2}{4}. \quad (26)$$

where we have the relations

$$\begin{aligned} \langle J_+^\lambda\rangle &= \overline{\langle J_-^\lambda\rangle} = \frac{\Gamma(\lambda + \frac{3}{2})}{\Gamma(\lambda + r - \frac{1}{2})} \bar{z} \\ &\times \frac{{}_0F_{2r-3}([], [\lambda + \frac{1}{2}, \lambda + \frac{3}{2}, \lambda + \frac{5}{2}, \lambda + \frac{5}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}{{}_1F_{2r-2}([\lambda + \frac{1}{2}], [\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}, \\ \langle J_+^{\lambda^2}\rangle &= \overline{\langle J_-^{\lambda^2}\rangle} = \frac{\Gamma(\lambda + \frac{3}{2})\Gamma(\lambda + \frac{5}{2})}{\Gamma(\lambda + r - \frac{1}{2})\Gamma(\lambda + r + \frac{1}{2})} \bar{z}^2 \\ &\times \frac{{}_1F_{2r-2}([\lambda + \frac{5}{2}], [\lambda + \frac{1}{2}, \lambda + \frac{3}{2}, \dots, \lambda + r - \frac{1}{2}, \lambda + r + \frac{1}{2}], |z|^2)}{{}_1F_{2r-2}([\lambda + \frac{1}{2}], [\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}, \\ \langle J_+^\lambda J_-^\lambda\rangle &= \frac{\Gamma(\lambda + \frac{3}{2})^2}{\Gamma(\lambda + r - \frac{1}{2})^2} |z|^2 \\ &\times \frac{{}_0F_{2r-3}([], [\lambda + \frac{1}{2}, \dots, \lambda + r - \frac{1}{2}, \lambda + r - \frac{1}{2}], |z|^2)}{{}_1F_{2r-2}([\lambda + \frac{1}{2}], [\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}, \\ \langle J_3^\lambda\rangle &= \left(\lambda + \frac{1}{2}\right) \\ &\times \frac{{}_2F_{2r-1}([\frac{\lambda}{2} + \frac{5}{4}], [\frac{\lambda}{2} + \frac{1}{4}, \lambda + \frac{1}{2}, \lambda + \frac{3}{2}, \lambda + \frac{3}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}{{}_1F_{2r-2}([\lambda + \frac{1}{2}], [\lambda + \frac{1}{2}, \lambda + \frac{1}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}. \end{aligned}$$

They result that, $\langle(\Delta X_{1(2)}^\lambda)^2\rangle$ as well as $S_{1(2)}^\lambda$, for any value of λ , are efficiently dependent on the complex variable $z(= |z|e^{i\varphi})$ and the deformation parameter r .

■ Case $r = 1$:

Our calculations show that the squeezing factor S_1^λ is really independent of λ . It illustrates that squeezing properties in the X_1 quadrature arise when φ is increased and culminates where φ reaches $\frac{\pi}{2}$ as well as $|z| \rightarrow 1$ (see figure 2(a)). However, figure 2(b) implies that squeezing properties in the X_2 quadrature is considerable where φ is decreased. It becomes maximal, $S_2^\lambda \rightarrow -1$, if φ tends to zero.

■ Case $r = 2$:

Clearly, for the case $r = 2$ we would not expect to take squeezing neither in X_1 nor in X_2 quadratures [46].

■ Case $r \geq 3$:

We show in figures 3(a_λ) the squeezing factor S_1^λ as a function of $|z|^2$ for different values of r ($= 3, 4$ and 5) as well as λ ($= -\frac{1}{4}, \frac{1}{4}, 1$), here we choose the phase $\varphi = 0$. They become really smaller than zero for any values of $|z|^2$. Where squeezing in the X_2 quadrature disappeared (figures 3(b_λ)). Figures 4(a_λ) show S_1^λ for different values of φ ($= 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$), while for fixed $r = 4$. We again find that by decreasing $|z|^2$ the degree of squeezing is enhanced when φ tends to zero. However we will loose the squeezing in X_1 when φ reaches $\frac{\pi}{4}$, where squeezing in the X_2 quadrature arises (see figures 4(b_λ)).

◇ *Anti-bunching effect and sub-Poissonian statistics*

Now we are in a position to study the anti-bunching effect as well as the statistics of GNLCs given by equation (7). We introduce the second-order correlation function for these states

$$(g^{(2)})^\lambda(|z|^2) = \frac{\langle \hat{N}^2 \rangle_r^\lambda - \langle \hat{N} \rangle_r^\lambda^2}{\langle \hat{N} \rangle_r^\lambda^2}, \quad (28)$$

furthermore, the inherent statistical properties of the GNCS follows also from calculating the Mandel parameter $Q_r^\lambda(|z|^2)^2$

$$Q_r^\lambda(|z|^2) = \langle \hat{N} \rangle_r^\lambda \left[(g^{(2)})^\lambda(|z|^2) - 1 \right]. \quad (29)$$

In order to find the function $(g^{(2)})^\lambda(|z|^2)$, also Mandel parameter $Q_r^\lambda(|z|^2)$, let us begin with the expectation values of the number operator \hat{N} and of the square of the number operator \hat{N}^2 in the basis of the Fock space states $|n, \lambda\rangle$

$$\begin{aligned} \langle \hat{N} \rangle_r^\lambda &= \frac{|z|^2}{\lambda + \frac{1}{2}} \left(\frac{\Gamma(\lambda + \frac{5}{2})}{\Gamma(\lambda + r + \frac{1}{2})} \right)^2 \frac{{}_0F_{2r-3}([], [\lambda + \frac{3}{2}, \dots, \lambda + r - \frac{1}{2}, \lambda + r - \frac{1}{2}], |z|^2)}{{}_0F_{2r-3}([], [\lambda + \frac{1}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}, \\ \langle \hat{N}^2 \rangle_r^\lambda &= \frac{|z|^2}{\lambda + \frac{1}{2}} \left(\frac{\Gamma(\lambda + \frac{3}{2})}{\Gamma(\lambda + r - \frac{1}{2})} \right)^2 \frac{{}_1F_{2r-2}([2], [1, \lambda + \frac{3}{2}, \dots, \lambda + r - \frac{1}{2}, \lambda + r - \frac{1}{2}], |z|^2)}{{}_0F_{2r-3}([], [\lambda + \frac{1}{2}, \dots, \lambda + r - \frac{3}{2}, \lambda + r - \frac{3}{2}], |z|^2)}. \end{aligned}$$

For the case $r = 1$, the second-order correlation function can be calculated to be taken as $(g^{(2)})^\lambda_{r=1}(|z|^2) = 1 + \frac{1}{1+\frac{\lambda}{2}} > 1$. This guaranties that $|z\rangle_1^\lambda$ exhibits a fully bunching effect, or

²A state for which $Q_r^\lambda(|z|^2) > 0$ (or $(g^{(2)})^\lambda_r(|z|^2) > 1$) is called super-Poissonian (bunching effect), if $Q = 0$ (or $g^{(2)} = 1$) the state is called Poissonian, while a state for which $Q < 0$ (or $g^{(2)} < 1$) is, also, called sub-Poissonian (antibunching effect).

super-Poissonian statistics. But this situation is changed when r and λ are enhanced. As shown in figure 5, $(g^{(2)})_r^\lambda(|z|^2)$ and $Q_r^\lambda(|z|^2)$ have been plotted in terms of $|z|^2$ for several values of $r(= 2, 3, 4 \text{ and } 5)$. These quantities are dependent on the analytical expressions of $\langle \hat{N} \rangle_r^\lambda$ and $\langle \hat{N}^2 \rangle_r^\lambda$ as functions of the variable $|z|^2$, deformation parameter r and λ . Because of the structure of these functions as illustrated in figure 5, the states GNLCs, $|z\rangle_{r \geq 2}^{\lambda=0}$, show sub-Poissonian statistics(or anti-bunching effect).

4 Conclusions

Based on the application of the generalized $su(1,1)$ displacement operators, broad range of states that are called generalized nonlinear coherent states are produced. In a general view the formalism presented here provides a unified approach to construct all the employed CSs already introduced in different ways(the Klauder-Perelomov and Barut-Girardello coherent states). These states realize a resolution of the identity with positive measures on the complex plane. Finally, non-classical properties of such states have been reviewed in detail. It has been shown that their squeezing properties are considerable and satisfy sub-Poissonian statistics, too. They lead to squeezing in one of X_1 or X_2 quadratures which could be varied not only by the deformation factor r , by the phase φ and λ . We assert that the above-mentioned approach can cover a wide range of nonlinear coherent states and be applied to some various, both physical and mathematical, models such as half oscillator and radial part of a 3D harmonic oscillator, too.

References

- [1] Schrodinger E 1926 Ann. Phys., Lpz 79 361
- [2] Glauber R J 1963 Phys. Rev 130 2529
- [3] Glauber R J 1963 Phys. Rev 131 2766
- [4] Klauder J R 1963 J. Math. Phys 4 1055
- [5] Klauder J R 1963 J. Math. Phys 4 1058
- [6] Sudarshan E C G 1963 Phys. Rev. Lett 10 277
- [7] Barut A O and Girardello L 1971 Commun. Math. Phys 21 41
- [8] Perelomov A M 1972 Commun. Math. Phys 26 222
- [9] Gazeau J P and Klauder J R 1999 J. Phys. A: Math. Gen 32 123
- [10] Antoine J P, Gazeau J P, Monceau P, Klauder J R and Penson K A 2001 J. Math. Phys 42 2349
- [11] Fukui T and Aizawa N 1993 Phys. Lett. A 180 308

- [12] Chenaghlou A and Fakhri H 2002 Mod. Phys. Lett. A 17 1701
- [13] Stoler D 1970 Phys. Rev. D 1 3217
- [14] Kimble H J, Dagenais M and Mandel L 1977 Phys. Rev. Lett 39 691
- [15] Short R and Mandel L 1983 Phys. Rev. Lett 51 384
- [16] Teich M C and Saleh B E A 1985 J. Opt. Soc. Am. B 2 275
- [17] Slusher R E, Hollberg L W, Yurke B, Mertz J C and Valley J F 1985 Phys. Rev. Lett 55 2409
- [18] Wu L -A , Kimble H J, Hall J L and Wu, H 1986 Phys. Rev. Lett 57 2520
- [19] Biedenharn L C 1989 J. Phys. A: Math. Gen 22 L873
- [20] Yurke B and Stoler D 1986 Phys. Rev. Lett 57 13
- [21] Noel M W and Stroud C R 1996 Jr. in Coherence and Quantum Optics VII Plenum, New York 563 564.
- [22] de Matos Filho R L and Vogel W 1996 Phys. Rev. Lett 76 608
- [23] Monroe C, Meekhof D M, King B E and Wineland D J 1996 Science 272 1131
- [24] Manko V I, Marmo G, Sudarshan E C G, Zaccaria F 1997 Phys. Scripta 55 528
- [25] de Matos Filho R L and Vogel W 1996 Phys. Rev. A 54 4560;
- [26] Raffa F. A, Rasetti M and Genovese M 2012 Phys. Lett. A 376 330
- [27] Kis Z, Vogel W and Davidovich L 2001 Phys. Rev. A 64 033401
- [28] Agarwal G S and Tara K 1991 Phys. Rev. A 43 492
- [29] Walls D F 1983 Nature 306 141
- [30] Hillery M 1987 Phys. Rev. A 36 3796; 1991 Phys. Rev. A 44 4578
- [31] Dehghani A and Mojaveri B 2012 J. Phys. A: Math. Theor 45 095304
- [32] Calogero F 1969 J. Math. Phys 10 2191
- [33] Perelemov A M 1986 Generalized Coherent States and Their Applications Springer, Berlin
- [34] Perelemov A M 1977 Sol. Phys. Usp. 20 703; 1972 Commun. Math. Phys 26 222
- [35] Agarwal G. S and Chaturvedi S J. Phys. A 1995 28 5747
- [36] Fakhri H, Dehghani A and Mojaveri B 2009 Int. J. Quantum Chem. 109 1228

- [37] Vourdas A 1990 Phys. Rev. A 41 1653
- [38] Zhang W -M, Feng D H and Gilmore R 1990 Rev. Mod. Phys 62 867
- [39] Wodkiewicz K and Eberly J H 1985 J. opt . Soc. Am. B 2 458K
- [40] Fu H C and Sasaki R 1996 Phys. Rev. A. 53 3836
- [41] Hall R L and Saad N (2000) J. Phys. A: Math. Gen. 33 569-578
- [42] Gradshteyn I S and Ryzhik I M 2000 Table of Integrals, Series, and Products (San Diego, CA: Academic)
- [43] Appl T and Schiller D H 2004 J. Phys. A: Math. Gen 37 2731
- [44] Cessa H M 2008 New expression for Laguerre and Hermite polynomials
arXiv:0809.2259v1 [math-ph]
- [45] Walls D F 1983 Nature 306 141
- [46] Buzek V 1990 J. Mod. Opt 37 303